Exercise Problems:

1) $$\int_0^3 \int_0^2 \int_0^1 (x+y+z)\, dz\, dy\, dx$$

Solution: $$\int_0^3 \int_0^2 \int_0^1 (x+y+z)\, dz\, dy\, dx.$$

$$= \int_0^3 \int_0^2 \left[\int_0^1 (x+y+z)\, dz \right] dy\, dx$$

$$= \int_0^3 \int_0^2 \left[xz + yz + \frac{z^2}{2} \right]_0^1 dy\, dx$$

$$= \int_0^3 \int_0^2 \left[x(1) + y(1) + \frac{1^2}{2} - 0 \right] dy\, dx$$

$$= \int_0^3 \left[\int_0^2 \left(x + y + \tfrac{1}{2}\right) dy \right] dx$$

$$= \int_0^3 \left[xy + \frac{y^2}{2} + \frac{y}{2} \right]_0^2 dx$$

$$= \int_0^3 \left[x(2) + \frac{2^2}{2} + \frac{2}{2} - 0 \right] dx$$

$$= \int_0^3 [2x + 2 + 1] \, dx$$

$$= \int_0^3 (2x+3) \, dx$$

$$= \left[2\frac{x^2}{2} + 3x \right]_0^3$$

$$= \left[x^2 + 3x \right]_0^3$$

$$= \left[3^2 + 3(3) - 0 \right]$$

$$= [9 + 9]$$

$$= 18 //$$

16) $\int_{-1}^{1} \int_{-2}^{2} \int_{-3}^{3} dx \, dy \, dz$

Solution: $\int_{-1}^{1} \int_{-2}^{2} \int_{-3}^{3} dx \, dy \, dz$

$$= \int_{-1}^{1} \int_{-2}^{2} [x]_{-3}^{3} \, dy \, dz$$

$$= \int_{-1}^{1} \int_{-2}^{2} [3 - (-3)] \, dy \, dz$$

$$= \int_{-1}^{1} \int_{-2}^{2} 6 \, dy \, dz$$

$$= \int_{-1}^{1} 6 \, [y]_{-2}^{2} \, dz$$

$$= \int_{-1}^{1} 6 \, [2+2] \, dz$$

$$= \int_{-1}^{1} (6 \times 4) \, dz$$

$$= \int_{-1}^{1} 24 \, dz$$

$$= [24z]_{-1}^{1}$$

$$= 24(1+1) = 24 \times 2 = 48$$

17) $$\int_0^1 \int_0^{1-x^2} \int_0^{1-x^2-y^2} dz\, dy\, dx$$

~~Solution:~~ $$\int_0^1 \int_0^{1-x} \int_0^{1-x^2-y^2} dz\, dy\, dx$$

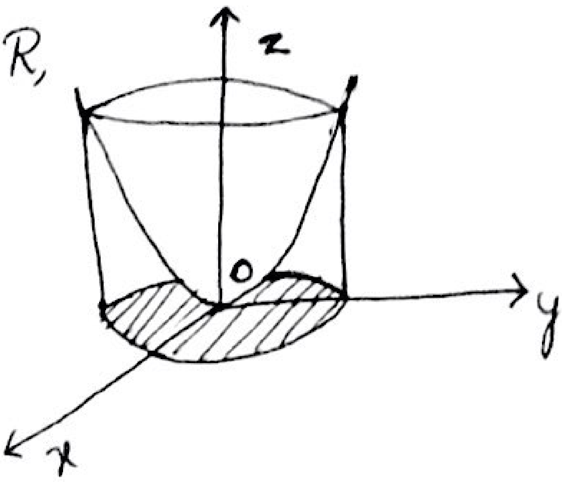
18) $$\int_0^1 \int_0^1 \int_{\sqrt{x^2+y^2}}^{2} xyz\, dz\, dy\, dx$$

19) $$\int_0^1 \int_0^{x^2} \int_0^{x+y} (x-2y+z)\, dz\, dy\, dx$$

20) $$\int_0^1 \int_0^{a-x} \int_0^{a-x-y} (x^2+y^2+z^2)\, dz\, dy\, dx$$

21) $$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2+z^2)}\, dx\, dy\, dz$$

28) If R is the region bounded by the cylinder $x^2+y^2=a^2$, the paraboloid $x^2+y^2=z$ and the xy-plane. Show that $\iiint_R z^2\,dx\,dy\,dz = \frac{\pi}{12}a^8$



__Solution:__ In the given region R, z varies from 0 to x^2+y^2.

For $z=0$, ~~x & y~~ (x,y) vary over the circular region bounded by the circle $x^2+y^2=a^2$.

i.e; for each x, y varies from $-\sqrt{a^2-x^2}$ to $\sqrt{a^2+x^2}$,

For $z=0$, $y=0$, x varies from $-a$ to a.

$$\therefore \iiint_R z^2\,dx\,dy\,dz = \int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{z=0}^{x^2+y^2} z^2\,dz\,dy\,dx$$

$$= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left[\frac{z^3}{3}\right]_0^{x^2+y^2} dy\, dx$$

$$= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left[\frac{(x^2+y^2)^3}{3} - 0\right] dy\, dx$$

$$= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{(x^2+y^2)^3}{3}\, dy\, dx$$

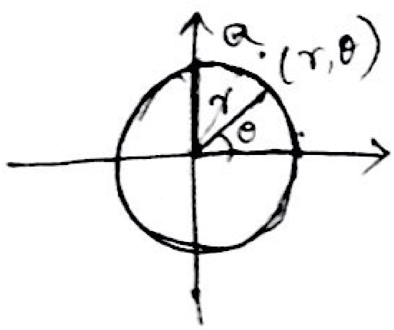
$$= \frac{1}{3} \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (x^2+y^2)^3\, dy\, dx$$

$\cancel{= \frac{1}{3}}$ Now the given area is bounded by the circle $x^2+y^2=a^2$

put $x = r\cos\theta$, $y = r\sin\theta$ } $x^2+y^2 = r^2$

$dy\,dx = r\,dr\,d\theta$

$$= \frac{1}{3} \int_{\theta=0}^{2\pi} \int_{r=0}^{a} r^6\, r\, dr\, d\theta$$



$$= \frac{1}{3} \int_{\theta=0}^{2\pi} \int_{r=0}^{a} r^7 \, dr \, d\theta$$

$$= \frac{1}{3} \int_{0}^{2\pi} \left[\frac{r^8}{8} \right]_0^a d\theta$$

$$= \frac{1}{3} \int_{0}^{2\pi} \left[\frac{a^8}{8} - 0 \right] d\theta$$

$$= \frac{1}{3} \int_{0}^{2\pi} \frac{a^8}{8} \, d\theta$$

$$= \frac{a^8}{24} \int_{0}^{2\pi} d\theta$$

$$= \frac{a^8}{24} \left[\theta \right]_0^{2\pi}$$

$$= \frac{a^8}{24} \left[2\pi - 0 \right]$$

$$= \frac{a^8}{\cancel{24}_{12}} \cancel{2}^{1}\pi = \frac{a^8 \pi}{12}$$

29. Evaluate $\iiint_R \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ if R is the region of integration bounded by the co-ordinate planes and the plane $x+y+z=1$.

Solution: The region of Integration is expressed as,

z varies from 0 to $1-x-y$

y varies from 0 to $1-x$

x varies from 0 to 1

$$\iiint_R \frac{dx\,dy\,dz}{(x+y+z+1)^3} = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} (x+y+z+1)^{-3}\,dz\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}\left[\frac{(x+y+z+1)^{-3+1}}{-3+1}\right]_0^{1-x-y} dy\,dx$$

$$= -\frac{1}{2}\int_0^1\int_0^{1-x}\left[(x+y+z+1)^{-2}\right]_0^{1-x-y} dy\,dx$$

$$= -\frac{1}{2}\int_0^1\int_0^{1-x}\left[(\cancel{x}+\cancel{y}+1-\cancel{x}-\cancel{y}+1)^{-2}-(x+y+1)^{-2}\right]dy\,dx$$

$$= -\frac{1}{2}\int_0^1\int_0^{1-x}\left[2^{-2}-(x+y+1)^{-2}\right]dy\,dx$$

$$= -\frac{1}{2}\int_0^1\int_0^{1-x}\left[1/4-(x+y+1)^{-2}\right]dy\,dx$$

$$= -\frac{1}{2}\int_0^1\left[y/4-\frac{(x+y+1)^{-2+1}}{-2+1}\right]_0^{1-x}dx$$

$$= -\frac{1}{2}\int_0^1\left[y/4-\frac{(x+y+1)^{-1}}{-1}\right]_0^{1-x}dx$$

$$= -\frac{1}{2}\int_0^1\left[y/4+\frac{1}{(x+y+1)}\right]_0^{1-x}dx$$

$$= -\frac{1}{2}\int_0^1\left[\frac{(1-x)}{4}+\frac{1}{(\cancel{x}+1-\cancel{x}+1)}\right]-\left[0+\frac{1}{x+1}\right]dx$$

$$= -\frac{1}{2}\int_0^1\left[\frac{(1-x)}{4}+\frac{1}{2}-\frac{1}{x+1}\right]dx$$

$$= -\frac{1}{2}\left[\frac{-(1-x)^2}{8} + \frac{x}{2} - \log(x+1)\right]_0^1$$

$$= -\frac{1}{2}\left[\cancel{-\frac{(1-1)^2}{8}} \to 0 + \frac{1}{2} - \log(2) - \left(-\frac{(1-0)^2}{8} + 0\right)\right]$$

$$= -\frac{1}{2}\left[\frac{1}{2} - \log(2) + \frac{1}{8}\right]$$

$$= -\frac{1}{2}\left[\frac{5}{8} - \log 2\right]$$

$$= \frac{1}{2}\log 2 - \frac{5}{16} //$$

H.W 30/ If R is the region bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$.

Show that $\iiint_R \frac{dx\,dy\,dz}{(1+x+y+z)^3} = \frac{1}{2}\left(\log 2 - \frac{5}{8}\right)$

31) Evaluate $\iiint_R (x+y+z)\,dx\,dy\,dz$ where R is the region bounded by the planes $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$

Solution: $\iiint_R (x+y+z)\,dx\,dy\,dz = \int_0^1\int_0^1\left[\int_0^1 (x+y+z)\,dx\right]dy\,dz$

$$= \int_0^1\int_0^1 \left[\frac{x^2}{2} + yx + zx\right]_0^1 dy\,dz$$

$$= \int_0^1\int_0^1 \left[\frac{1^2}{2} + y(1) + z(1) - 0\right] dy\,dz$$

$$= \int_0^1\int_0^1 \left[1/2 + y + z\right] dy\,dz$$

$$= \int_0^1 \left[y/2 + y^2/2 + zy\right]_0^1 dz$$

$$= \int_0^1 \left[1/2 + 1^2/2 + z(1) - 0\right] dz$$

$$= \int_0^1 \left[\tfrac{1}{2} + \tfrac{1}{2} + z \right] dz$$

$$= \int_0^1 (1+z)\, dz$$

$$= \left[z + \frac{z^2}{2} \right]_0^1$$

$$= \left[1 + \frac{1^2}{2} - 0 \right]$$

$$= \left[1 + \tfrac{1}{2} \right] = 3/2$$

If R is the region bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$, Show that

$$\iiint_R z\, dx\, dy\, dz = \frac{1}{24}$$

<u>solution:</u> The region of integration R is expressed as, z varies from 0 to $1-x-y$

y varies from 0 to $1-x$

x varies from 0 to 1